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An addition formula for the Jacobian theta function and its applications

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Abstract

In this paper, we prove an addition formula for the Jacobian theta function using the theory of elliptic functions. It turns out to be a fundamental identity in the theory of theta functions and elliptic function, and unifies many important results about theta functions and elliptic functions. From this identity we can derive the Ramanujan cubic theta function identity, Winquist's identity, a theta function identities with five parameters, and many other interesting theta function identities; and all of which are as striking as Winquist's identity. This identity allows us to give a new proof of the addition formula for the Weierstrass sigma function. A new identity about the Ramanujan cubic elliptic function is given. The proofs are self contained and elementary.

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1. Introduction

We suppose throughout this paper that $q = \exp(2\pi i\tau)$, where τ has positive imaginary part. We will use the familiar notation

$$(z; q)_{\infty} = \prod_{n=0}^{\infty} (1 - zq^n), \quad (1.1)$$

and sometimes write

$$(a, b, c, \dots; q)_{\infty} = (a; q)_{\infty} (b; q)_{\infty} (c; q)_{\infty} \cdots. \quad (1.2)$$

To carry out our study, we need some basic facts about the Jacobian theta function $\theta_1(z|\tau)$ which is defined as

$$\begin{aligned} \theta_1(z|\tau) &= -iq^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{(2n+1)iz} \\ &= 2q^{1/8} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z \end{aligned} \quad (1.3)$$

(see, for example, [24, p. 463]). From this we readily find that

$$\theta_1(z + \pi|\tau) = -\theta_1(z|\tau) \quad \text{and} \quad \theta_1(z + \pi\tau|\tau) = -q^{-1/2} e^{-2iz} \theta_1(z|\tau). \quad (1.4)$$

Using the well-known Jacobian triple product identity

$$(q, z, q/z; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n \quad (1.5)$$

(see [1, pp. 21–22], [3, p. 35], [10,15]), we can deduce the infinite product representation for $\theta_1(z|\tau)$, namely,

$$\begin{aligned} \theta_1(z|\tau) &= 2q^{1/8} (\sin z) (q, qe^{2iz}, qe^{-2iz}; q)_{\infty} \\ &= iq^{1/8} e^{-iz} (q, e^{2iz}, qe^{-2iz}; q)_{\infty} \end{aligned} \quad (1.6)$$

(see, for example, [24, p. 469]). In this paper we use $\theta'_1(z|\tau)$ to denote the partial derivative of $\theta_1(z|\tau)$ with respect to z . Differentiating the first equation in (1.6) with respect to z and then putting $z = 0$ gives

$$\theta'_1(0|\tau) = 2q^{1/8} (q; q)_{\infty}^3. \quad (1.7)$$

Now we state the principal theta function identity of this paper, which is the starting point of our investigation.

Theorem 1. If $h_1(z|\tau)$ and $h_2(z|\tau)$ are two entire functions of z , which satisfying the functional equations

$$h(z + \pi|\tau) = -h(z|\tau) \quad \text{and} \quad h(z + \pi\tau|\tau) = -q^{-3/2}e^{-6iz}h(z|\tau). \quad (1.8)$$

Then there is a constant C independent of x and y such that

$$\begin{aligned} & (h_1(x|\tau) - h_1(-x|\tau))(h_2(y|\tau) - h_2(-y|\tau)) \\ & - (h_2(x|\tau) - h_2(-x|\tau))(h_1(y|\tau) - h_1(-y|\tau)) \\ & = C\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau). \end{aligned} \quad (1.9)$$

This identity may be viewed as an addition formula for the Jacobian theta function $\theta_1(z|\tau)$. To the best knowledge of the author, it never appeared in the literature. This is a very important theta function identity; and it implies many interesting special cases. From which, we can derive Winkist's identity, the Ramanujan cubic theta function identity and many other interesting theta function identities. This identity also allows us to give a new proof of the addition formula for the Weierstrass sigma function.

The rest of the paper are organized as follows. In Section 2, we prove Theorem 1 using the classical theory of elliptic functions. In Section 3, we derive Winkist's identity, the Ramanujan cubic theta function identity, a theta function identities with five parameters, and some other interesting theta function identities from Theorem 1. In Section 4, we give a new proof of the addition formula for the Weierstrass sigma function based on Theorem 1. Following Ramanujan, for each k , we define

$$S_{2k}(\tau) := \sum_{n=1}^{\infty} \frac{n^{2k}q^n}{1+q^n+q^{2n}} \quad (1.10)$$

and let $L(z|\tau)$ be the generating function of $S_{2k}(\tau)$ defined by

$$L(z|\tau) = \sum_{k=1}^{\infty} (-1)^k S_{2k}(\tau) \frac{(2z)^{2k}}{(2k)!}. \quad (1.11)$$

It is known that $L(z|\tau)$ is an elliptic function with periods π and $3\pi\tau$ [9]; and in this paper we call it as the Ramanujan cubic elliptic function. In Section 5, we first prove the following identity by employing addition formula for the Weierstrass sigma function.

Theorem 2. Let $\theta_1(z|\tau)$ be the Jacobian theta function and $L(z|\tau)$ be defined as in (1.11). Then we have

$$L(z|\tau) = -\frac{\theta_1^3(z|3\tau)}{2\theta_1(z|\tau)}. \quad (1.12)$$

Then we apply this identity to study the Ramanujan cubic theory of elliptic functions.

2. The proof of Theorem 1

Now we begin to prove Theorem 1.

Proof. For brevity we temporarily denote

$$\delta_1(z) = h_1(z|\tau) - h_1(-z|\tau) \quad \text{and} \quad \delta_2(z) = h_2(z|\tau) - h_2(-z|\tau).$$

Then the identity in (1.9) can be written as

$$\delta_1(x)\delta_2(y) - \delta_2(x)\delta_1(y) = C\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau). \quad (2.1)$$

Let

$$H(x) := \delta_1(x)\delta_2(y) - \delta_2(x)\delta_1(y)$$

and

$$G(x) := \theta_1(x|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau).$$

Using (1.8) we can verify that $H(x)$ satisfies the functional equations

$$f(x+\pi) = -f(x) \quad \text{and} \quad f(x+\pi\tau) = -q^{-3/2}e^{-6ix}f(x). \quad (2.2)$$

Using (1.4) we readily find that $G(x)$ also satisfies the functional equations in (2.2). It follows that $H(x)/G(x)$ is an elliptic function with periods π and $3\pi\tau$. We assume at the moment that $0 < y < \pi$. Then $G(x)$ would have zeros at $x = 0, y$, and $\pi - y$ in the period parallelogram; and all of which are simple zero points. By using (1.8) and direct computations, we find that $x = 0, y$, and $\pi - y$ are also zero points of $H(x)$ in the period parallelogram. Thus $H(x)/G(x)$ has no poles in the period parallelogram and hence $H(x)/G(x)$ must be a constant. This constant is independent of x , but dependent of y . We denote this constant by $C(y)$; then we have $H(x) = C(y)G(x)$, or

$$\delta_1(x)\delta_2(y) - \delta_2(x)\delta_1(y) = C(y)\theta_1(x|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau). \quad (2.3)$$

We interchange x and y to obtain

$$\delta_1(x)\delta_2(y) - \delta_2(x)\delta_1(y) = C(x)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau). \quad (2.4)$$

Comparing the above two equations we immediately have

$$C(y)\theta_1(x|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau) = C(x)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau). \quad (2.5)$$

It follows that

$$C(y)\theta_1(x|\tau) = C(x)\theta_1(y|\tau). \quad (2.6)$$

Thus we have

$$\frac{C(x)}{\theta_1(x|\tau)} = \frac{C(y)}{\theta_1(y|\tau)}. \quad (2.7)$$

This identity indicates that $C(x)/\theta_1(x|\tau)$ is independent of x and so it must be a constant, say C . Thus we have $C(y) = C\theta_1(y|\tau)$. Substituting this back to (2.3), we arrive at (2.1). This completes the proof of Theorem 1. \square

3. The applications of Theorem 1 to the theta functions identities

3.1. Winquist's identity

In this section we first prove the following theta function identity using Theorem 1, which is an variant form of Winquist's identity.

Theorem 3. *We have*

$$\begin{aligned} & q^{1/4}(q; q)_{\infty}^2 \theta_1(3y|3\tau) (e^{2ix} \theta_1(3x + \pi\tau|3\tau) + e^{-2ix} \theta_1(3x - \pi\tau|3\tau)) \\ & - q^{1/4}(q; q)_{\infty}^2 \theta_1(3x|3\tau) (e^{2iy} \theta_1(3y + \pi\tau|3\tau) + e^{-2iy} \theta_1(3y - \pi\tau|3\tau)) \\ & = \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau). \end{aligned} \quad (3.1)$$

Proof. Taking $h_1(z|\tau) = e^{2iz} \theta_1(3z + \pi\tau|3\tau)$ and $h_2(z|\tau) = \theta_1(3z|3\tau)$ in Theorem 1, then (1.9) becomes

$$\begin{aligned} & 2\theta_1(3y|3\tau) (e^{2ix} \theta_1(3x + \pi\tau|3\tau) + e^{-2ix} \theta_1(3x - \pi\tau|3\tau)) \\ & - 2\theta_1(3x|3\tau) (e^{2iy} \theta_1(3y + \pi\tau|3\tau) + e^{-2iy} \theta_1(3y - \pi\tau|3\tau)) \\ & = C\theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau). \end{aligned} \quad (3.2)$$

To determine C , we set $y = \pi/3$ in the above equation. After using the fact $\theta_1(\pi|3\tau) = 0$ and some simple calculations we obtain

$$\begin{aligned} & 4i \left(\sin \frac{2\pi}{3} \right) \theta_1(\pi\tau|3\tau) \theta_1(3x|3\tau) \\ & = C\theta_1\left(\frac{\pi}{3}|\tau\right) \theta_1(x|\tau) \theta_1\left(x + \frac{\pi}{3}|\tau\right) \theta_1\left(x - \frac{\pi}{3}|\tau\right). \end{aligned} \quad (3.3)$$

Dividing both sides by x and then letting $x \rightarrow 0$ yields

$$12i \left(\sin \frac{2\pi}{3} \right) \theta_1(\pi\tau|3\tau) \theta_1'(0|3\tau) = -C\theta_1'(0|\tau) \theta_1^3\left(\frac{\pi}{3}|\tau\right). \quad (3.4)$$

Appealing to the infinite representation for $\theta_1(z|\tau)$, we can find

$$\theta_1\left(\frac{\pi}{3}|\tau\right) = \sqrt{3}q^{1/8}(q^3; q^3)_{\infty} \quad \text{and} \quad \theta_1(\pi\tau|3\tau) = iq^{-1/8}(q; q)_{\infty}. \quad (3.5)$$

Substituting

$$\theta_1'(0|\tau) = 2q^{\frac{1}{8}}(q; q)_{\infty}^3, \quad \theta_1'(0|3\tau) = 2q^{\frac{3}{8}}(q^3; q^3)_{\infty}^3, \quad \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2},$$

and (3.5) into (3.4) we find

$$C = 2q^{-1/4}(q; q)_{\infty}^{-2}. \quad (3.6)$$

Set back this into (3.2), we arrive at (3.1). This completes the proof of the theorem. \square

The Winquist identity is the following identity.

Theorem 4. *We have*

$$\begin{aligned} & \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{\frac{1}{2}(3m^2+3n^2+3m+n)} \\ & \times \{x^{-3m}y^{-3n} - x^{-3m}y^{3n+1} - x^{-3n+1}y^{-3m-1} + x^{3n+2}y^{-3m-1}\} \\ & = (q; q)_{\infty}^2 (x, q/x, y, q/y, xy, q/xy, x/y, qy/x; q)_{\infty}. \end{aligned} \quad (3.7)$$

Proof. From the series expansion for $\theta_1(z|\tau)$ we have

$$\theta_1(3z|3\tau) = -iq^{3/8} \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{3}{2}m(m+1)} e^{(6m+3)iz}. \quad (3.8)$$

In the same way we have

$$e^{2iz}\theta_1(3z + \pi\tau|3\tau) = iq^{-\frac{1}{8}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(3n^2+n)} e^{-(6n+1)iz}. \quad (3.9)$$

Replacing z by $-z$ we have

$$e^{-2iz}\theta_1(3z - \pi\tau|3\tau) = -iq^{-\frac{1}{8}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(3n^2+n)} e^{(6n+1)iz}. \quad (3.10)$$

Thus the left side of (3.1) can be written as

$$\begin{aligned} & q^{1/2}(q; q)_{\infty}^2 \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{\frac{1}{2}(3m^2+3n^2+3m+n)} \{e^{(6m+3)ix+(6n+1)iy} - e^{(6m+3)ix-(6n+1)iy} \\ & + e^{(6m+3)iy-(6n+1)ix} - e^{(6m+3)iy+(6n+1)ix}\}. \end{aligned} \quad (3.11)$$

Using the infinite product expansion formula for $\theta_1(z|\tau)$, we find the right side of (3.1) can be written as

$$q^{1/2} e^{i(3x+y)} (q; q)_\infty^2 (e^{-2ix}, qe^{2ix}, e^{-2iy}, qe^{2iy}; q)_\infty \\ \times (e^{-2i(x+y)}, qe^{2i(x+y)}, e^{-2i(x-y)}, qe^{2i(x-y)}; q)_\infty. \quad (3.12)$$

Equating the above two equations and then writing e^{-2ix} as x and e^{-2iy} as y gives (3.7). This completes the proof of the theorem. \square

Identity (3.7) is first established by L. Winquist in [25]; and since then many proofs have appeared [8,11,13–15]. By his identity (3.7), Winquist derive the identity

$$48(q; q)_\infty^{10} = \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} (6m+3)(6n+1) \\ \times \{(6m+3)^2 - (6n+1)^2\} q^{\frac{1}{2}(3m^2+3n^2+3m+n)} \quad (3.13)$$

and from which he give a simple proof of Ramanujan's partition congruence for the modulus 11, $p(11n+6) \equiv 0 \pmod{11}$, where $p(n)$ denotes the number of unrestricted partitions of the positive integer n .

3.2. The Ramanujan cubic theta function identity

Theorem 5. Let $\theta_1(x|\tau)$ be the Jacobian theta function; and let $a(\tau)$ be the Ramanujan function (see [4]) defined by

$$a(\tau) := 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}} \right). \quad (3.14)$$

Then, we have

$$\theta_1^3\left(x + \frac{\pi}{3} \middle| \tau\right) + \theta_1^3\left(x - \frac{\pi}{3} \middle| \tau\right) - \theta_1^3(x|\tau) = 3a(\tau)\theta_1(3x|3\tau) \quad (3.15)$$

and

$$\theta_1^3(x|3\tau) - q^{1/2} e^{2ix} \theta_1^3(x + \pi\tau|3\tau) - q^{1/2} e^{-2ix} \theta_1^3(x - \pi\tau|3\tau) = a(\tau)\theta_1(x|\tau). \quad (3.16)$$

Equality (3.16) is the Ramanujan cubic theta function identity [5, p. 142, Entry 3]. The above two identities can also found in [20, p. 827]. Now we prove the above two identities for cubic theta functions using Theorem 1.

Proof. Differentiating both sides of (1.9) with respect to y , then setting $y = 0$, and finally writing $C\theta'_1(0|\tau)/2$ as C , we conclude that

$$h'_2(0|\tau)(h_1(x|\tau) - h_1(-x|\tau)) - h'_1(0|\tau)(h_2(x|\tau) - h_2(-x|\tau)) = C\theta_1^3(x|\tau). \quad (3.17)$$

Using (1.4) we can verify that $\theta_1(3z|3\tau)$ and $\theta_1^3(z + \frac{\pi}{3}|\tau)$ satisfy all the conditions of Theorem 1. So we can take $h_1(z) = \theta_1(3z|3\tau)$ and $h_2(z) = \theta_1^3(z + \frac{\pi}{3}|\tau)$. Then the above equation becomes

$$\begin{aligned}
 & 2h'_2(0|\tau)\theta_1(3x|3\tau) - h'_1(0|\tau)\left(\theta_1^3\left(x + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(x - \frac{\pi}{3}|\tau\right)\right) \\
 & = C\theta_1^3(x|\tau).
 \end{aligned} \tag{3.18}$$

Now we begin to compute $h'_1(0|\tau)$ and $h'_2(0|\tau)$. Using (1.7) we find

$$h'_1(0|\tau) = 3\theta'_1(0|3\tau) = 6q^{3/8}(q^3; q^3)_\infty^3. \tag{3.19}$$

To compute $h'_2(0|\tau)$, we need to know the values of $\theta_1(\frac{\pi}{3}|\tau)$ and $\theta'_1(\frac{\pi}{3}|\tau)$. Appealing to the infinite product representation for $\theta_1(z|\tau)$, we readily find that

$$\theta_1\left(\frac{\pi}{3}|\tau\right) = \theta_1\left(\frac{2\pi}{3}|\tau\right) = \sqrt{3}q^{1/8}(q^3; q^3)_\infty. \tag{3.20}$$

In this paper we use the notation $\frac{\theta'_1}{\theta_1}(z|\tau)$ to denote the logarithmic derivative of $\theta_1(z|\tau)$. It is well known that

$$\frac{\theta'_1}{\theta_1}(z|\tau) = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin 2nz \tag{3.21}$$

(see [24, p. 489]). Thus, we have

$$\frac{\theta'_1}{\theta_1}\left(\frac{\pi}{3}|\tau\right) = \frac{1}{\sqrt{3}}a(\tau). \tag{3.22}$$

It follows that

$$h'_2(0|\tau) = 3\theta_1^3\left(\frac{\pi}{3}|\tau\right)\left(\frac{\theta'_1}{\theta_1}\right)\left(\frac{\pi}{3}|\tau\right) = 9q^{3/8}(q^3; q^3)_\infty^3 a(\tau). \tag{3.23}$$

Substituting (3.19) and (3.23) into (3.18) yields

$$\begin{aligned}
 C\theta_1^3(x|\tau) &= 18q^{3/8}a(\tau)(q^3; q^3)_\infty^3 \theta_1(3x|3\tau) \\
 &\quad - 6q^{3/8}(q^3; q^3)_\infty^3 \left(\theta_1^3\left(z + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(z - \frac{\pi}{3}|\tau\right)\right).
 \end{aligned} \tag{3.24}$$

Setting $z = \frac{\pi}{3}$ and using the facts that $\theta_1(\pi|3\tau) = 0$ and $\theta_1(\frac{\pi}{3}|\tau) = \theta_1(\frac{2\pi}{3}|\tau)$, we arrive at

$$C\theta_1^3\left(\frac{\pi}{3}|\tau\right) = -6q^{3/8}(q^3; q^3)_\infty^3 \theta_1^3\left(\frac{\pi}{3}|\tau\right). \tag{3.25}$$

It follows that

$$C = -6q^{3/8}(q^3; q^3)_\infty^3. \tag{3.26}$$

We substituting this back to (3.24) and then cancel the common factors to arrive at (3.14). Proceeding through the same steps as before, by taking $h_1(z|\tau) = \theta_1(z|\frac{\tau}{3})$ and $h_2(z|\tau) = e^{2iz}\theta_1^3(z + \frac{\pi\tau}{3}|\tau)$ in Theorem 1, we can obtain

$$\begin{aligned} \theta_1^3(x|\tau) - q^{1/6}e^{2ix}\theta_1^3\left(x + \frac{\pi\tau}{3}|\tau\right) - q^{1/6}e^{-2ix}\theta_1^3\left(x - \frac{\pi\tau}{3}|\tau\right) \\ = a\left(\frac{\tau}{3}\right)\theta_1\left(x\left|\frac{\tau}{3}\right.\right). \end{aligned} \quad (3.27)$$

Replacing q by q^3 gives (3.15). Thus, we complete the proof of the theorem. \square

3.3. One identity of the author in [20]

In [20], the author prove the following striking identity. Here we reprove it using Theorem 1.

Theorem 6. *We have*

$$\begin{aligned} q^{\frac{1}{12}}(q; q)_{\infty}^2\theta_1\left(x\left|\frac{\tau}{3}\right.\right)\theta_1(3y|3\tau) - q^{\frac{1}{12}}(q; q)_{\infty}^2\theta_1\left(y\left|\frac{\tau}{3}\right.\right)\theta_1(3x|3\tau) \\ = \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau). \end{aligned} \quad (3.28)$$

Proof. Taking $h_1(z|\tau) = \theta_1(z|\frac{\tau}{3})$ and $h_2(z|\tau) = \theta_1(3z|3\tau)$, then (1.9) becomes

$$\begin{aligned} 2\theta_1\left(x\left|\frac{\tau}{3}\right.\right)\theta_1(3y|3\tau) - 2\theta_1\left(y\left|\frac{\tau}{3}\right.\right)\theta_1(3x|3\tau) \\ = C\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau). \end{aligned} \quad (3.29)$$

Taking $y = \frac{\pi}{3}$ and using the fact $\theta_1(\pi|3\tau) = 0$, we immediately have

$$-2\theta_1\left(\frac{\pi}{3}\left|\frac{\tau}{3}\right.\right)\theta_1(3x|3\tau) = C\theta_1\left(\frac{\pi}{3}|\tau\right)\theta_1(x|\tau)\theta_1\left(x + \frac{\pi}{3}|\tau\right)\theta_1\left(x - \frac{\pi}{3}|\tau\right). \quad (3.30)$$

Dividing both sides by x and then letting $x \rightarrow 0$ yields

$$-6\theta_1\left(\frac{\pi}{3}\left|\frac{\tau}{3}\right.\right)\theta_1'(0|3\tau) = -C\theta_1'(0|\tau)\theta_1^3\left(\frac{\pi}{3}|\tau\right). \quad (3.31)$$

Substituting

$$\begin{aligned} \theta_1\left(\frac{\pi}{3}\left|\frac{\tau}{3}\right.\right) &= q^{1/(24)}\sqrt{3}(q; q)_{\infty}, & \theta_1\left(\frac{\pi}{3}|\tau\right) &= q^{1/8}\sqrt{3}(q^3; q^3)_{\infty}, \\ \theta_1'(0|\tau) &= 2q^{1/8}(q; q)_{\infty}^3, & \theta_1'(0|3\tau) &= 2q^{3/8}(q^3; q^3)_{\infty}^3 \end{aligned}$$

into (3.31) and after simplifying gives

$$C = 2q^{-1/(12)}(q; q)_{\infty}^{-2}. \quad (3.32)$$

Set back this into (3.29), we can obtain (3.28). This completes the proof of the theorem. \square

In [20] the identity (3.28) has been used to give the following new representation for $(q; q)_{\infty}^{10}$:

$$32(q; q)_{\infty}^{10} = 9 \left(\sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^3 q^{\frac{3}{2}n(n+1)} \right) \left(\sum_{n=-\infty}^{\infty} (2n+1) q^{\frac{1}{6}n(n+1)} \right) \\ - \left(\sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{\frac{3}{2}n(n+1)} \right) \left(\sum_{n=-\infty}^{\infty} (2n+1)^3 q^{\frac{1}{6}n(n+1)} \right). \quad (3.33)$$

Using this identity, the authors [7] have also given a short proof of Ramanujan's famous congruence $p(11n+6) \equiv 0 \pmod{11}$. This identity also has also been used to derive some identities for the Hirschhorn–Garvan–Borwein two variable cubic theta functions [12].

3.4. A theta function with five parameters

Theorem 7. *We have*

$$\theta_1(x-u|\tau)\theta_1(x+u|\tau)\theta_1(y-v|\tau)\theta_1(y+v|\tau) \\ - \theta_1(x-v|\tau)\theta_1(x+v|\tau)\theta_1(y-u|\tau)\theta_1(y+u|\tau) \\ = \theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(u-v|\tau)\theta_1(u+v|\tau). \quad (3.34)$$

Proof. For brevity we temporarily denote $\theta_1(z) := \theta_1(z|\tau)$. Taking

$$h_1(z|\tau) = \theta_1(z)\theta_1(z-u)\theta_1(z+u) \quad \text{and} \quad h_2(z|\tau) = \theta_1(z)\theta_1(z-v)\theta_1(z+v)$$

in Theorem 1 yields

$$4\theta_1(x)\theta_1(y)\theta_1(x-u)\theta_1(x+u)\theta_1(y-v)\theta_1(y+v) \\ - 4\theta_1(x)\theta_1(y)\theta_1(x-v)\theta_1(x+v)\theta_1(y-u)\theta_1(y+u) \\ = C\theta_1(x)\theta_1(y)\theta_1(x-y)\theta_1(x+y). \quad (3.35)$$

We cancel the factor $\theta_1(x)\theta_1(y)$ to get

$$4\theta_1(x-u)\theta_1(x+u)\theta_1(y-v)\theta_1(y+v) \\ - 4\theta_1(x-v)\theta_1(x+v)\theta_1(y-u)\theta_1(y+u) \\ = C\theta_1(x-y)\theta_1(x+y). \quad (3.36)$$

Setting $x = v$, we immediately have

$$C = 4\theta_1(u-v)\theta_1(u+v). \quad (3.37)$$

Substituting this back to (3.35), we deduce that (3.34). This completes the proof of the theorem. \square

When $v = 0$, (3.34) reduces to

$$\begin{aligned} & \theta_1(x - u|\tau)\theta_1(x + u|\tau)\theta_1^2(y|\tau) - \theta_1(y - u|\tau)\theta_1(y + u|\tau)\theta_1^2(x|\tau) \\ &= \theta_1(x - y|\tau)\theta_1(x + y|\tau)\theta_1^2(u|\tau). \end{aligned} \quad (3.38)$$

This identity has important applications to Number theory. In [18], using this identity, logarithmic differentiation, and series manipulation, we give a simple method to compute the number of representations of the positive integers n as a sum of k triangular numbers.

3.5. A list of theta function identities

By taking $h_1(z|\tau) = \theta_1^3(z|\tau)$ and $h_2(z|\tau) = \theta_1(3z|3\tau)$; and $h_1(z|\tau) = \theta_1^3(z|\tau)$ and $h_2(z|\tau) = \theta_1(z|\frac{\tau}{3})$ respectively in Theorem 1, we can find the following two identities [20, p. 827].

Theorem 8. *We have*

$$\begin{aligned} & (q; q)_\infty^3 \theta_1^3(x|\tau)\theta_1(3y|3\tau) - (q; q)_\infty^3 \theta_1^3(y|\tau)\theta_1(3x|3\tau) \\ &= 3q^{1/4}(q^3; q^3)_\infty^3 \theta_1(x + y|\tau)\theta_1(x - y|\tau)\theta_1(x|\tau)\theta_1(y|\tau) \end{aligned} \quad (3.39)$$

and

$$\begin{aligned} & (q; q)_\infty^3 \theta_1^3(x|\tau)\theta_1\left(y\left|\frac{\tau}{3}\right.\right) - (q; q)_\infty^3 \theta_1^3(y|\tau)\theta_1\left(x\left|\frac{\tau}{3}\right.\right) \\ &= q^{-1/(12)}(q^{1/3}; q^{1/3})_\infty^3 \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x + y|\tau)\theta_1(x - y|\tau). \end{aligned} \quad (3.40)$$

Similarly, by taking $h_1(z|\tau) = \theta_1^3(z|\tau)$ and $h_1(z|\tau) = \theta_1^3(z + \frac{\pi}{3}|\tau)$ in Theorem 1, we have the identity

$$\begin{aligned} & (q; q)_\infty^3 \theta_1^3(x|\tau) \left\{ \theta_1^3\left(y + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(y - \frac{\pi}{3}|\tau\right) \right\} \\ & - (q; q)_\infty^3 \theta_1^3(y|\tau) \left\{ \theta_1^3\left(x + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(x - \frac{\pi}{3}|\tau\right) \right\} \\ &= 9q^{1/4}a(\tau)(q^3; q^3)_\infty^3 \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x + y|\tau)\theta_1(x - y|\tau), \end{aligned} \quad (3.41)$$

where $a(\tau)$ are the Ramanujan function defined as in (3.14).

Taking $h_1(z|\tau) = \theta_1^3(z|\tau)$ and $h_1(z|\tau) = e^{2iz}\theta_1^3(z + \frac{\pi\tau}{3}|\tau)$ in Theorem 1, we find

$$\begin{aligned} & (q; q)_\infty^3 \theta_1^3(x|\tau) \left\{ e^{2iy}\theta_1^3\left(y + \frac{\pi\tau}{3}|\tau\right) + e^{-2iy}\theta_1^3\left(y - \frac{\pi\tau}{3}|\tau\right) \right\} \\ & - (q; q)_\infty^3 \theta_1^3(y|\tau) \left\{ e^{2ix}\theta_1^3\left(x + \frac{\pi\tau}{3}|\tau\right) + e^{-2ix}\theta_1^3\left(x - \frac{\pi\tau}{3}|\tau\right) \right\} \\ &= -q^{-1/4}a\left(\frac{\tau}{3}\right)(q^{1/3}; q^{1/3})_\infty^3 \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x + y|\tau)\theta_1(x - y|\tau). \end{aligned} \quad (3.42)$$

Taking $h_1(z|\tau) = \theta_1(3z|3\tau)$ and $h_1(z|\tau) = \theta_1^3(z + \frac{\pi}{3}|\tau)$ in Theorem 1 yields

$$\begin{aligned} & (q; q)_\infty^3 \theta_1(3y|3\tau) \left\{ \theta_1^3\left(x + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(x - \frac{\pi}{3}|\tau\right) \right\} \\ & - (q; q)_\infty^3 \theta_1(3x|3\tau) \left\{ \theta_1^3\left(y + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(y - \frac{\pi}{3}|\tau\right) \right\} \\ & = 3q^{1/4} (q^3; q^3)_\infty^3 \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau). \end{aligned} \quad (3.43)$$

Taking $h_1(z|\tau) = \theta_1(z|\frac{\tau}{3})$ and $h_1(z|\tau) = e^{2iz} \theta_1^3(z + \frac{\pi\tau}{3}|\tau)$ in Theorem 1 gives

$$\begin{aligned} & (q; q)_\infty^3 \theta_1\left(y|\frac{\tau}{3}\right) \left\{ e^{2ix} \theta_1^3\left(x + \frac{\pi\tau}{3}|\tau\right) + e^{-2ix} \theta_1^3\left(x - \frac{\pi\tau}{3}|\tau\right) \right\} \\ & - (q; q)_\infty^3 \theta_1\left(x|\frac{\tau}{3}\right) \left\{ e^{2iy} \theta_1^3\left(y + \frac{\pi\tau}{3}|\tau\right) + e^{-2iy} \theta_1^3\left(y - \frac{\pi\tau}{3}|\tau\right) \right\} \\ & = q^{-1/4} (q^{1/3}; q^{1/3})_\infty^3 \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau). \end{aligned} \quad (3.44)$$

4. The addition formula for the Weierstrass sigma function

In this section we will use Theorem 1 to give a new proof of the addition formula for the Weierstrass elliptic function $\wp(z|\tau)$. We begin this section with the following lemma ([19, p. 720], [21, p. 8]).

Lemma 9. *Let $\theta_1(z|\tau)$ be the Jacobian theta function defined in (1.3). Then we have*

$$\begin{aligned} \frac{\theta_1'}{\theta_1}(z|\tau) &= \frac{1}{z} - \frac{1}{3} E_2(\tau) z - \frac{1}{45} E_4(\tau) z^3 - \frac{2}{945} E_6(\tau) z^5 + \cdots \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} E_{2k}(\tau) z^{2k-1}. \end{aligned} \quad (4.1)$$

Here B_k are the Bernoulli numbers defined as the coefficients in the power series

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}, \quad |x| < 2\pi, \quad (4.2)$$

and $E_{2k}(\tau)$ are the normalized Eisenstein series defined by

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n}. \quad (4.3)$$

The first three $E_{2k}(\tau)$ are

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \quad (4.4)$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad (4.5)$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}. \quad (4.6)$$

We use $\wp(z|\tau)$ to represent the Weierstrass \wp function of periods π and $\pi\tau$. The Weierstrass \wp function is related to the Jacobian theta function $\theta_1(z|\tau)$ by the relation

$$\begin{aligned} \wp(z|\tau) &= \csc^2 - 8 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \cos 2nz - \frac{1}{3} E_2(\tau) \\ &= -\frac{1}{3} E_2(\tau) - \left(\frac{\theta_1'}{\theta_1} \right)'(z|\tau) \end{aligned} \quad (4.7)$$

(see, for example, [24, p. 460]). By this formula we can use the Jacobian theta functions to study the Weierstrass function $\wp(z|\tau)$; and this is a clever choice in many cases. The Weierstrass $\sigma(z|\tau)$ function and the Jacobian theta function $\theta_1(z|\tau)$ satisfy the identity

$$\sigma(z|\tau) = \exp(\eta z^2) \frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}. \quad (4.8)$$

The addition formula for the Weierstrass sigma function is

$$\wp(x|\tau) - \wp(y|\tau) = \frac{\sigma(x+y|\tau)\sigma(x-y|\tau)}{\sigma^2(x|\tau)\sigma^2(y|\tau)}. \quad (4.9)$$

This is a fundamental identity in the theory of elliptic functions (see, for example, [24, p. 325, Eq. (1.7)]). Using the above two equations we find that (4.9) can be written as

Theorem 10. *Let $\theta_1(z|\tau)$ be defined as in (1.3). Then we have*

$$\left(\frac{\theta_1'}{\theta_1} \right)'(x|\tau) - \left(\frac{\theta_1'}{\theta_1} \right)'(y|\tau) = \theta_1'(0|\tau)^2 \frac{\theta_1(x-y|\tau)\theta_1(x+y|\tau)}{\theta_1^2(x|\tau)\theta_1^2(y|\tau)}. \quad (4.10)$$

Here we will use Theorem 1 to give a very simple proof of it.

Proof. By taking $h_2(z) = \theta_1^3(z|\tau)$ in Theorem 1 and after simplification, we find (1.9) becomes

$$\begin{aligned} &\theta_1'(0|\tau)(h_1(x|\tau) - h_1(-x|\tau))\theta_1^3(y|\tau) \\ &\quad - \theta_1'(0|\tau)(h_1(y|\tau) - h_1(-y|\tau))\theta_1^3(x|\tau) \\ &\quad = -2h_1'(0|\tau)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau). \end{aligned} \quad (4.11)$$

From Lemma 9 we find near $z = 0$ that

$$\left(\frac{\theta'_1}{\theta_1}\right)'(z|\tau) = -\frac{1}{z^2} + O(1). \quad (4.12)$$

Using (1.4) we can check that $\left(\frac{\theta'_1}{\theta_1}\right)'(z|\tau)$ is an elliptic function with periods π and $\pi\tau$. Thus we find that

$$\theta_1^3(z|\tau) \left(\frac{\theta'_1}{\theta_1}\right)'(z|\tau)$$

satisfies the functional equations in (1.8). Taking

$$h_1(z|\tau) = \theta_1^3(z|\tau) \left(\frac{\theta'_1}{\theta_1}\right)'(z|\tau)$$

in (4.11); and noting that $\theta_3(z|\tau)$ is odd function of z and $\left(\frac{\theta'_1}{\theta_1}\right)'(z|\tau)$ is even function of z , we find that

$$\left(\frac{\theta'_1}{\theta_1}\right)'(x|\tau) - \left(\frac{\theta'_1}{\theta_1}\right)'(y|\tau) = -\frac{h'_1(0|\tau)}{\theta'_1(0|\tau)} \frac{\theta_1(x-y|\tau)\theta_1(x+y|\tau)}{\theta_1^2(x|\tau)\theta_1^2(y|\tau)}. \quad (4.13)$$

Using (4.12) we readily find $h_1(0|\tau) = -\theta'_1(0|\tau)^3$. Substituting this into (4.13). We immediately have (4.10). This completes the proof of the theorem. \square

Sometimes we may write (4.10) in the form

$$\wp(x|\tau) - \wp(y|\tau) = -\theta'_1(0|\tau)^2 \frac{\theta_1(x+y|\tau)\theta_1(x-y|\tau)}{\theta_1^2(x|\tau)\theta_1^2(y|\tau)}. \quad (4.14)$$

5. Ramanujan's cubic theory of elliptic functions

We begin this section by proving Theorem 2.

Proof. Logarithmic differentiation to the second equation in (1.6) gives

$$\frac{\theta'_1}{\theta_1}(z|\tau) = -i - 2i \sum_{n=0}^{\infty} \frac{q^n e^{2iz}}{1 - q^n e^{2iz}} + 2i \sum_{n=1}^{\infty} \frac{q^n e^{-2iz}}{1 - q^n e^{-2iz}}. \quad (5.1)$$

It follows that [22]

$$\frac{\theta'_1}{\theta_1}(z + \pi\tau|3\tau) - \frac{\theta'_1}{\theta_1}(z - \pi\tau|3\tau) = -2i - 4i \sum_{n=1}^{\infty} \frac{q^n \cos 2nz}{1 + q^n + q^{2n}}. \quad (5.2)$$

Setting $z = 0$ yields

$$2\frac{\theta'_1}{\theta_1}(\pi\tau|3\tau) = -2i - 4i \sum_{n=1}^{\infty} \frac{q^n}{1+q^n+q^{2n}} = -\frac{2}{3}(2+a(\tau))i. \quad (5.3)$$

Subtracting (5.3) from (5.2) we obtain

$$\frac{\theta'_1}{\theta_1}(z+\pi\tau|3\tau) - \frac{\theta'_1}{\theta_1}(z-\pi\tau|3\tau) - 2\frac{\theta'_1}{\theta_1}(\pi\tau|3\tau) = -4i \sum_{n=1}^{\infty} \frac{q^n(\cos 2nz - 1)}{1+q^n+q^{2n}}. \quad (5.4)$$

Substituting the series

$$\cos 2nz - 1 = \sum_{k=1}^{\infty} (-1)^k \frac{(2nz)^{2k}}{(2k)!} \quad (5.5)$$

into the right side of the above equation, interchanging the order of summation, we find that

$$\frac{\theta'_1}{\theta_1}(z+\pi\tau|3\tau) - \frac{\theta'_1}{\theta_1}(z-\pi\tau|3\tau) - 2\frac{\theta'_1}{\theta_1}(\pi\tau|3\tau) = -4iL(z|\tau). \quad (5.6)$$

Recall the identity in (4.10)

$$\left(\frac{\theta'_1}{\theta_1}\right)'(x|\tau) - \left(\frac{\theta'_1}{\theta_1}\right)'(y|\tau) = \theta'_1(0|\tau)^2 \frac{\theta_1(x-y|\tau)\theta_1(x+y|\tau)}{\theta_1^2(x|\tau)\theta_1^2(y|\tau)}. \quad (5.7)$$

Dividing both sides by $x-y$ and then letting $y \rightarrow x$, we have

$$\left(\frac{\theta'_1}{\theta_1}\right)''(x|\tau) = \theta'_1(0|\tau)^3 \frac{\theta_1(2x|\tau)}{\theta_1^4(x|\tau)}. \quad (5.8)$$

Differentiating both sides of (5.7) with respect to x , we have

$$\begin{aligned} \left(\frac{\theta'_1}{\theta_1}\right)''(x|\tau) &= \theta'_1(0|\tau)^2 \frac{\theta_1(x-y|\tau)\theta_1(x+y|\tau)}{\theta_1^2(x|\tau)\theta_1^2(y|\tau)} \\ &\quad \times \left(\frac{\theta'_1}{\theta_1}(x+y|\tau) + \frac{\theta'_1}{\theta_1}(x-y|\tau) - 2\frac{\theta'_1}{\theta_1}(x|\tau) \right). \end{aligned} \quad (5.9)$$

Comparing (5.8) and (5.9) we conclude that

$$\frac{\theta'_1}{\theta_1}(x+y|\tau) + \frac{\theta'_1}{\theta_1}(x-y|\tau) - 2\frac{\theta'_1}{\theta_1}(x|\tau) = \frac{\theta'_1(0|\tau)\theta_1(2x|\tau)\theta_1^2(y|\tau)}{\theta_1^2(x|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)}. \quad (5.10)$$

Using the identity $(z, zq, zq^2; q^3)_\infty = (z; q)_\infty$, and the infinite product representation for $\theta_1(z|\tau)$, we readily find that

$$\theta_1(z|\tau) = \frac{(q; q)_\infty}{(q^3; q^3)_\infty^3} \theta_1(z|3\tau) \theta_1(z + \pi\tau|3\tau) \theta_1(z - \pi\tau|3\tau) \quad (5.11)$$

and

$$\theta_1(\pi\tau|3\tau) = iq^{-1/8}(q; q)_\infty \quad \text{and} \quad \theta_1(2\pi\tau|3\tau) = iq^{-5/8}(q; q)_\infty. \quad (5.12)$$

Replacing y by z and τ by 3τ in (5.10) and then taking $x = \pi\tau$, and finally using (5.11) and (5.12) in the resulting equation, we obtain

$$\frac{\theta'_1}{\theta_1}(z + \pi\tau|3\tau) - \frac{\theta'_1}{\theta_1}(z - \pi\tau|3\tau) - 2\frac{\theta'_1}{\theta_1}(\pi\tau|3\tau) = 2i \frac{\theta_1^3(z|3\tau)}{\theta_1(z|\tau)}. \quad (5.13)$$

Comparing (5.6) and the above equation we obtain (1.12). This completes the proof of Theorem 2. The identity in [23, Eq. (6.1)] is essentially equivalent to (1.12). \square

Let

$$z_3(\tau) := a(\tau) = 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \quad (5.14)$$

and

$$x_3(\tau) := \frac{c^3(\tau)}{a^3(\tau)}, \quad \text{where } c(\tau) = 3q^{1/3} \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty}. \quad (5.15)$$

Then we will use Theorem 2 to prove the following identity about the Weierstrass elliptic function \wp and $L(z|\tau)$ [9].

Theorem 11. *Let $\wp(z|3\tau)$ be the Weierstrass elliptic function with periods π and $3\pi\tau$; and $L(z|\tau)$ be defined as in (1.11). Then we have*

$$\wp(z|3\tau) = -\frac{2}{27} \frac{x_3 z_3^3}{L(z|\tau)} - \frac{z_3^2}{3} \quad (5.16)$$

where $x_3 := x_3(\tau)$ and $z_3 := z_3(\tau)$.

Proof. Replacing τ by 3τ in (4.14) and then setting $y = \pi\tau$, we have

$$\wp(x|3\tau) - \wp(\pi\tau|3\tau) = -\theta'_1(0|3\tau)^2 \frac{\theta_1(x + \pi\tau|3\tau) \theta_1(x - \pi\tau|3\tau)}{\theta_1^2(x|\tau) \theta_1^2(\pi\tau|3\tau)}. \quad (5.17)$$

Applying (5.11) and (5.12) in the right side of the above equation, we arrive at

$$\wp(x|3\tau) - \wp(\pi\tau|3\tau) = 4q \frac{(q^3; q^3)_\infty^9}{(q; q)_\infty^3} \frac{\theta_1(x|\tau)}{\theta_1^3(x|3\tau)} = \frac{4}{27} x_3 z_3^3 \frac{\theta_1(x|\tau)}{\theta_1^3(x|3\tau)}. \quad (5.18)$$

Comparing (1.12) with the above equation we have

$$\wp(z|3\tau) = -\frac{2}{27} \frac{x_3 z_3^3}{L(z|\tau)} + \wp(\pi\tau|3\tau). \quad (5.19)$$

Next we begin to compute $\wp(\pi\tau|3\tau)$. Logarithmic differentiation to (5.11) gives

$$\frac{\theta'_1}{\theta_1}(z + \pi\tau|3\tau) + \frac{\theta'_1}{\theta_1}(z - \pi\tau|3\tau) = \frac{\theta'_1}{\theta_1}(z|\tau) - \frac{\theta'_1}{\theta_1}(z|3\tau). \quad (5.20)$$

Using (4.1) we find that

$$\frac{\theta'_1}{\theta_1}(z|\tau) - \frac{\theta'_1}{\theta_1}(z|3\tau) = \frac{1}{3}(E_2(3\tau) - E_2(\tau))z + O(z^3). \quad (5.21)$$

Equating coefficients of z on the left side of (5.20) and the right side of (5.21) yields

$$\left(\frac{\theta'_1}{\theta_1}\right)'(\pi\tau|3\tau) = \frac{1}{6}(E_2(3\tau) - E_2(\tau)). \quad (5.22)$$

Combining (4.7) and the above equation we have

$$\wp(\pi\tau|3\tau) = -\frac{1}{3}E_2(3\tau) - \left(\frac{\theta'_1}{\theta_1}\right)'(\pi\tau|3\tau) = -\frac{1}{2}E_3(3\tau) + \frac{1}{6}E_2(\tau). \quad (5.23)$$

From [17, Eq. (1.36), p. 152] we know that

$$a^2(\tau) = \frac{3}{2}E_3(3\tau) - \frac{1}{2}E_2(\tau). \quad (5.24)$$

It follows that

$$\wp(\pi\tau|3\tau) = -\frac{1}{3}a^2(\tau) = -\frac{1}{3}z_3^2. \quad (5.25)$$

Substituting this into (5.19) gives (5.16), which completes the proof of the theorem. \square

Recall that the elliptic function $\wp(z|3\tau)$ satisfies the differential equation (see, for example, [2, p. 11]).

$$[\wp'(z|3\tau)]^2 = 4\wp^3(z|3\tau) - \frac{4}{3}E_4(3\tau)\wp(z|3\tau) - \frac{8}{27}E_6(3\tau). \quad (5.26)$$

We recall the following identities of Ramanujan [6,16]:

$$E_4(3\tau) = z_3^4 \left(1 - \frac{8}{9}x_3\right) \quad \text{and} \quad E_6(3\tau) = z_3^6 \left(1 - \frac{4}{3}x_3 + \frac{8}{27}x_3^3\right). \quad (5.27)$$

Substituting (5.16) and (5.27) into (5.26) we find the following differential equation satisfied by $L(z|\tau)$ [6,9]:

$$[L'(z|\tau)]^2 = -\frac{8}{27}z_3^3x_3L(z|\tau) - 4z_3^2L^2(z|\tau) - 16z_3L^3(z|\tau) - 16L^4(z|\tau). \quad (5.28)$$

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